Equation of a Straight Line in 2D and 3D

In 2D, a line is most commonly described by an equation of the form $y = mx + c$, this is known as the Cartesian form in 2D space. There is an alternative way to describe lines, the vector form:

$$\boldsymbol{r} = \boldsymbol{a} + \lambda \boldsymbol{b}$$

Where $\boldsymbol{a}$ is a vector from the origin to a point on the line, $\boldsymbol{b}$ is a vector parallel to the line and λ is a free parameter which has a specific value for each point on the line.

The advantage of the vector equation for a line is that it easily generalises to 3D, and higher dimensions. The equation has the exact same form in higher dimensions; the only difference is that the vectors themselves are also of a higher dimension (i.e. in 2D the vectors $\boldsymbol{r}$, $\boldsymbol{a}$ and $\boldsymbol{b}$ have two components, in 3D they have three components). Given any two points in 3D, the equation of the line on which they lie can be found by setting $\boldsymbol{a}$ equal to one of the points, and $\boldsymbol{b}$ equal to the vector going between the points. There is always freedom in what to make $\boldsymbol{b}$, since it can be of any length.

Having written the vector equation of a line in 3D, the Cartesian form of the equation for the same line can be written in terms of the components of $\boldsymbol{a}$ and $\boldsymbol{b}$:

$$\frac{x - a_x}{b_x} = \frac{y - a_y}{b_y} = \frac{z - a_z}{b_z} = \lambda$$

This comes from re-arranging each component of the vector equation for λ and setting them all equal to each other.

Example 1: Find the equation of the line which contains the points $(3, 2, 4)$ and $(4, 6, 8)$ in both vector and Cartesian form.

Choosing $\boldsymbol{a} = \begin{pmatrix}3\\2\\4\end{pmatrix}$, and calculating $\boldsymbol{b}$.	$\boldsymbol{a} = \begin{pmatrix}3\\2\\4\end{pmatrix}, \boldsymbol{b} = \begin{pmatrix}3\\2\\4\end{pmatrix} - \begin{pmatrix}4\\6\\8\end{pmatrix} = \begin{pmatrix}-1\\-4\\-4\end{pmatrix}$
The vector equation of the line can now be written.	$\boldsymbol{r} = \begin{pmatrix}3\\2\\4\end{pmatrix} + \lambda \begin{pmatrix}-1\\-4\\-4\end{pmatrix}$
Using the formula above, the cartesian form can be written and simplified.	$\frac{x-3}{-1} = \frac{y-2}{-4} = \frac{z-4}{-4}$ $\Rightarrow \frac{x-3}{1} = \frac{y-2}{4} = \frac{z-4}{4}$

Scalar Product (Dot Product)

There is a way of combining two vectors to produce a scalar called the scalar or dot product. There are two equivalent formulae for the dot product:

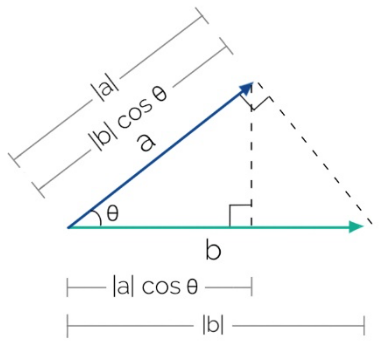
$$\boldsymbol{a} \cdot \boldsymbol{b} = a_x b_x + a_y b_y + a_z b_z = |\boldsymbol{a}||\boldsymbol{b}| \cos\theta$$

This can be used to find the angle between two vectors, and thus two lines, although care must be taken as this formula gives the angle between vectors with tails touching.

If $\boldsymbol{a}$ and $\boldsymbol{b}$ are orthogonal then:

$$\boldsymbol{a} \cdot \boldsymbol{b} = 0$$

The dot product has a useful geometrical interpretation – a projection of $\boldsymbol{a}$ onto $\boldsymbol{b}$ followed by a scaling by $|\boldsymbol{b}|$. Similarly, it can be seen as a projection of $\boldsymbol{b}$ onto $\boldsymbol{a}$ followed by a scaling of $|\boldsymbol{a}|$.



Equation of a plane (A Level Only)

A plane can also be described using a vector equation:

$$\boldsymbol{r} = \boldsymbol{a} + \lambda \boldsymbol{b} + \mu \boldsymbol{c}$$

As before, $\boldsymbol{a}$ is a vector to the plane, $\boldsymbol{b}$ and $\boldsymbol{c}$ are two non-parallel vectors along the plane, λ and μ parameters which have a unique value for each point on the plane.

Given a plane, a normal vector to the plane, n can be found and used to give the equation of the plane in cartesian form:

$$n_x x + n_y y + n_z z = d$$

Where d is a constant which can be found using the co-ordinates of any point known to be on the plane. If the normal vector is chosen to have unit length, then d is the shortest distance between the plane and origin. This can also be written more succinctly using the dot product:

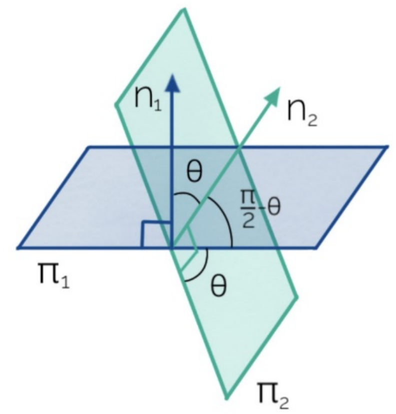
$$\boldsymbol{r} \cdot \boldsymbol{n} = d$$

Geometry of Planes and Lines (A Level Only)

Common questions include finding the angle between two planes or a line and a plane. When finding an angle involving a plane, it is usually easier to find the angle made between the plane normals first. In these types of problems, there is always an acute and an obtuse angle. To find the acute angle, take the modulus of the dot product.

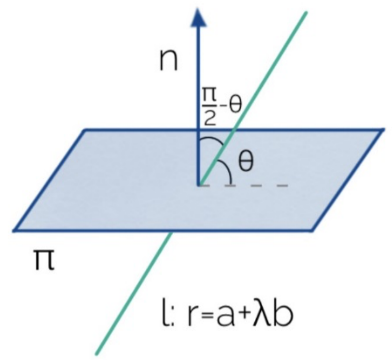
Angle between two planes, π_1 and π_2, with normals $\boldsymbol{n_1}$ and $\boldsymbol{n_2}$:

$$\cos\theta = \left|\frac{\boldsymbol{n_1} \cdot \boldsymbol{n_2}}{|\boldsymbol{n_1}||\boldsymbol{n_2}|}\right|$$



Angle between a line, ℓ which is parallel to $\boldsymbol{b}$ and a plane with normal $\boldsymbol{n}$:

$$\cos\frac{\pi}{2} - \theta = \sin\theta = \left|\frac{\boldsymbol{b} \cdot \boldsymbol{n}}{|\boldsymbol{b}||\boldsymbol{n}|}\right|$$



Example 2: **a.)** Find the vector equation of the plane containing the points $(0, 4, 2)$, $(1, 2, 0)$ and $(1, 1, 1)$. **b.)** Find the cartesian equation for the same plane.

a.) First, two vectors parallel to the plane are calculated.	$\begin{pmatrix}1\\2\\0\end{pmatrix} - \begin{pmatrix}1\\1\\1\end{pmatrix} = \begin{pmatrix}0\\1\\-1\end{pmatrix}, \begin{pmatrix}0\\4\\2\end{pmatrix} - \begin{pmatrix}1\\1\\1\end{pmatrix} = \begin{pmatrix}-1\\3\\1\end{pmatrix}$
These two vectors (since they are not parallel) and one of the points given in the question are sufficient for writing the vector equation for the plane.	$\boldsymbol{r} = \begin{pmatrix}1\\1\\1\end{pmatrix} + \lambda \begin{pmatrix}0\\1\\-1\end{pmatrix} + \mu \begin{pmatrix}-1\\3\\1\end{pmatrix}$
b.) Now, the normal vector must be orthogonal to both vectors along the plane. This gives simultaneous equations for the normal vector. (There is a quicker way to do this using the vector product which hasn't been introduced yet.)	$\begin{pmatrix}n_x\\n_y\\n_z\end{pmatrix} \cdot \begin{pmatrix}0\\1\\-1\end{pmatrix} = 0 \Rightarrow n_y - n_z = 0 \quad (1)$ $\begin{pmatrix}n_x\\n_y\\n_z\end{pmatrix} \cdot \begin{pmatrix}-1\\3\\1\end{pmatrix} = 0 \Rightarrow -n_x + 3n_y + n_z = 0 \quad (2)$
There are three unknowns but only two equations, so there are infinite solutions. This is expected as there are infinitely many normal vectors (varying only in length not direction). We use the two equations to find simpler relations between the components of the vector.	$(1) \Rightarrow n_y = n_z$ $\Rightarrow 4n_y = n_x$ $\Rightarrow n_y = n_z = \frac{n_x}{4}$
Picking the n_x component to be 4 gives a valid normal vector, and the cartesian equation can be found. The point (1,1,1) is used to determine the constant d.	$\boldsymbol{n} = \begin{pmatrix}4\\1\\1\end{pmatrix} \Rightarrow 4x + y + z = 6$

Example 3: A line L is given by $\frac{x}{2} - 1 = y + 3 = 1 - \frac{z}{2}$, a plane P is given by the equation $x + y + z = 1$. Find the acute angle between L and P.

First, to use the formula above, the vector form of the equation for L is needed.	$\lambda = \frac{x}{2} - 1 = y + 2 = 1 - \frac{z}{2} \Rightarrow x = 2\lambda + 2, y = \lambda - 3, z = 2 - 2\lambda$ $\boldsymbol{r} = \begin{pmatrix}2\\-3\\2\end{pmatrix} + \lambda \begin{pmatrix}2\\1\\-2\end{pmatrix}$
From the equation of P, the normal vector can be seen easily. Now using the vector parallel to the line and the normal vector for the plane, the angle can be found using the formula $\sin\theta = \left\lvert\frac{\boldsymbol{b}\cdot\boldsymbol{n}}{\lvert\boldsymbol{b}\rvert\lvert\boldsymbol{n}\rvert}\right\rvert$.	$\boldsymbol{n} = \begin{pmatrix}1\\1\\1\end{pmatrix}, \boldsymbol{b} = \begin{pmatrix}2\\1\\-2\end{pmatrix} \Rightarrow \boldsymbol{b} \cdot \boldsymbol{n} = 2 + 1 - 2 = 1, \lvert\boldsymbol{b}\rvert = 3, \lvert\boldsymbol{n}\rvert = \sqrt{3}$ $\sin\theta = \left\lvert\frac{1}{3\sqrt{3}}\right\rvert = \frac{1}{3\sqrt{3}} \Rightarrow \theta = 0.194 \text{ rad}$